Longevity improvement, retirement age and aggregate savings

by

Yanyou Chen and Sau-Him Paul Lau

9 Supplementary Material

9.1 Parameters of the pay-as-you-go system

In the model, a person starts to work at (adult) age 0, until retirement at age $R$, which will be determined endogenously. Define $N(s,x)$ as the labor supply (either 1 or 0) of a cohort $s$ individual at age $x$, which is given by

$$N(s,x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq R \\
0 & \text{otherwise} 
\end{cases}. \quad (34)$$

Define $B(t)$ as the number of birth at time $t$. Since $x = t - s$, we have $dx = -ds$. Therefore, aggregate labor supply at time $t$, $N(t)$, is given by

$$N(t) = \int_{t-R}^{t} B(s) l(t-s) N(s,t-s) ds = \int_{t-T}^{t} B(t) e^{-n(t-s)} l(t-s) N(s,t-s) ds$$

$$= B(t) \int_{0}^{T} e^{-nx} l(x) N(s,x) dx = B(t) \int_{0}^{R} e^{-nx} l(x) dx. \quad (35)$$

Assume a pay-as-you-go system with pure transfer. At each period, each worker pays a fraction ($\tau$) of his wage income. On the other hand, each retiree obtains a fraction $b$, the replacement ratio, of the market wage (in real term) when he retires.

At current time $t$, individuals who were born from time $t-R$ to $t$ (i.e. age $R$ or younger) are working and pay social security taxes, and individuals who were born from $t-T$ to $t-R$ will collect social security benefit. Consider the steady-state equilibrium. Denote a variable at the steady-state equilibrium with a *. Balanced budget of the social security system implies that

$$\tau w^*(t) \int_{t-R}^{t} B(s) l(t-s) ds = \int_{t-T}^{t-R} B(s) l(t-s) bw^*(s+R) ds \quad (36)$$

where $w^*(t)$ is the wage rate at time $t$ along the steady-state equilibrium, given in (7). The above equation implies that $b$ and $\tau$ are related according to:

$$\tau w^*(t) \int_{0}^{R} B(t-x) l(x) dx = b \int_{R}^{T} B(t-x) l(x) w^*(t-x+R) dx$$
9.2 Derivation of the steady state equilibrium

An individual born at time \( s \) chooses the future consumption path, at time \( t \) (where \( s \leq t \leq s + T \)), to maximize (1), subject to the flow budget constraint

\[
\frac{\partial a(s, x)}{\partial x} = [r(s + x) + \mu(x)]a(s, x)
\]

\[
+ (1 - \tau) w(s + x) N(s, x) + bw(s + R)[1 - N(s, x)] - c(s, x),
\]

and the conditions \( a(s, 0) = a(s, T) = 0 \). His lifetime budget constraint is given by

\[
\int_0^T e^{-rxl(x)} c^*(s, x, R) dx
\]

\[
= (1 - \tau) \int_0^R e^{-rxl(x)} w^*(s + x) dx + bw^*(s + R) \int_R^T e^{-rxl(x)} dx
\]

(a) The first step is to find the first-order conditions for optimal consumption and retirement age.

The procedure in obtaining the Keynes-Ramsey rule for this intertemporal consumption problem is standard. It is given by (8).

For optimal retirement age, we substitute (8) into (1) to express the objective function in terms of \( R \) only. Denote it by \( V(R) \). Differentiating it with respect to \( R \), we obtain

\[
V'(R) = \int_0^T e^{-pxl(x)} \frac{1}{c^*(s, x, R)} \frac{\partial c^*(s, x, R)}{\partial R} dx - e^{-\rho Rl(R) \phi(R)}
\]

\[
= \int_0^T e^{-pxl(x)} \frac{1}{e^{(r-\rho)x}c^*(s, 0, R)} \frac{\partial c^*(s, x, R)}{\partial R} dx - e^{-\rho Rl(R) \phi(R)}
\]

\[
= \frac{1}{c^*(s, 0, R)} \int_0^T e^{-rxl(x)} \frac{\partial c^*(s, x, R)}{\partial R} dx - e^{-\rho Rl(R) \phi(R)}
\]

Differentiating the lifetime budget constraint with respect to \( R \) (and using \( \frac{dw^*(s + R)}{dR} = gw^*(s + R) \)), we obtain

\[
\int_0^T e^{-rxl(x)} \frac{\partial c^*(s, x, R)}{\partial R} dx = (1 - \tau) e^{-R^*l(R) w^*(s + R)}
\]
\[ +bgw^* (s + R) \int_R^T e^{-r^*x} l(x) \, dx - bw^* (s + R) e^{-r^*R} l(R) \]

\[
- \int_0^T e^{-r^*x} l(x) \frac{\partial c^*(s, x, R)}{\partial R} \, dx
\]

\[= (1 - \tau - b) e^{-r^*R} l(R) w^* (s + R) + bgw^* (s + R) \int_R^T e^{-r^*x} l(x) \, dx\]

Therefore,

\[
V'(R) = \frac{1}{e^*(s, 0, R)} \left[ (1 - \tau - b) e^{-r^*R} l(R) + bg \int_R^T e^{-r^*x} l(x) \, dx \right] w^* (s + R) - e^{-\rho R} l(R) \phi (R)
\]

The first-order condition is

\[
c^*(s, 0, R^*)^{-1} \left[ (1 - \tau - b) + bg \int_R^T e^{-r^*x} l(x) \, dx \right] e^{-r^*R^*} w(s + R^*) = e^{-\rho R^*} \phi(R^*).
\]

(The second-order condition holds under standard assumptions.)

(b) Next, express the consumption level at the beginning of adult life, \( c^*(s, 0, R^*) \), in terms of other variables

\[
\int_0^T e^{-r^*x} l(x) c^*(s, x, R^*) \, dx
\]

\[= (1 - \tau) \int_0^{R^*} e^{-r^*x} l(x) w^* (s + x) \, dx + bw^* (s + R^*) \int_{R^*}^T e^{-r^*x} l(x) \, dx \]

\[
- \int_0^T e^{-r^*x} l(x) e^{g^x} c^*(s, 0, R^*) \, dx
\]

\[= (1 - \tau) \int_0^{R^*} e^{-r^*x} l(x) e^{g(s+x)} w^* (0) \, dx + be^{g(s+R^*)} w^* (0) \int_{R^*}^T e^{-r^*x} l(x) \, dx \]

\[
- c^*(s, 0, R^*) \int_0^T e^{-\rho x} l(x) \, dx
\]

\[= e^{g^s} w^* A(0) \left[ (1 - \tau) \int_0^{R^*} e^{-r^*x} l(x) e^{g^x} \, dx + be^{gR^*} \int_{R^*}^T e^{-r^*x} l(x) \, dx \right]
\]

\[
- c^*(s, 0, R^*) = e^{g^s} w^* A(0) \frac{(1 - \tau) \int_0^{R^*} e^{-(r^*-g)x} l(x) \, dx + be^{gR^*} \int_{R^*}^T e^{-r^*x} l(x) \, dx}{\int_0^T e^{-\rho x} l(x) \, dx}
\]
Thus, the first-order condition for retirement age becomes

\[
\begin{align*}
    c^*(s, 0, R^*)^{-1} \left[ (1 - \tau - b) + bg \frac{\int_{R^*}^T e^{-r^*x} l(x) dx}{e^{-r^*R^*} l(R^*)} \right] e^{-r^*R^*} w^*(s + R^*) &= e^{-\rho R^*} \phi(R^*) \\
    \rightarrow \left( \frac{1}{e^{\gamma s} w^* A(0)} \right) \frac{\int_0^T e^{-\rho x} l(x) dx}{(1 - \tau) \int_0^T e^{-(r^*-\gamma)x} l(x) dx + be^{\rho R^*} \int_{R^*}^T e^{-r^*x} l(x) dx} & \times \left[ (1 - \tau - b) + bg \frac{\int_{R^*}^T e^{-r^*x} l(x) dx}{e^{-r^*R^*} l(R^*)} \right] e^{-r^* R^*} e^{g(s+R^*)} w^* A(0) = e^{-\rho R^*} \phi(R^*) \\
    e^{-(r^*-\gamma)R^*} \left[ (1 - \tau - b) + bg \frac{\int_{R^*}^T e^{-r^*x} l(x) dx}{e^{-r^*R^*} l(R^*)} \right] & \times \int_0^T e^{-\rho x} l(x) dx (1 - \tau) \int_0^T e^{-(r^*-\gamma)x} l(x) dx + be^{\rho R^*} \int_{R^*}^T e^{-r^*x} l(x) dx = e^{-\rho R^*} \phi(R^*).
\end{align*}
\]

(42)

(c) Next, it can be shown that aggregate consumption at the steady-state equilibrium is given by

\[
C^* (t) = \int_{t-T}^t B (s) l(t - s) c^*(s, t - s, R^*) ds
\]

\[
= \int_{t-T}^t B (t) e^{-n(t-s)} l(t - s) e^{(r^*-\rho)(t-s)} c^*(s, 0, R^*) ds
\]

\[
= \int_{t-T}^t B (t) e^{-n(t-s)} l(t - s) e^{(r^*-\rho)(t-s)} e^{\gamma s} c^*(0, 0, R^*) ds
\]

\[
= e^{\gamma t} B (t) \int_0^T e^{-(g+n-r^*+\rho)x} l(x) c^*(0, 0, R^*) dx
\]

\[
= e^{\gamma t} B (t) c^*(0, 0, R^*) \int_0^T e^{-(g+n-r^*+\rho)x} l(x) dx
\]

(43)

Therefore,

\[
c^* = \frac{C^* (t)}{A(t) N (t)}
\]

\[
w^* \left( \frac{1 - \tau}{\int_0^T e^{-(r^*-\gamma)x} l(x) dx + be^{\rho R^*} \int_{R^*}^T e^{-r^*x} l(x) dx} \right) \left( \frac{\int_0^T e^{-(g+n-r^*+\rho)x} l(x) dx}{\int_0^T e^{-\rho x} l(x) dx} \right).
\]

(44)

Using methods similar to d’Albis (2007), the equation that determines capital per worker at the steady-state equilibrium is

\[
k^* = \frac{w^*}{r^* - g - n}
\]

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\[
\left[ \frac{(1 - \tau) \int_{0}^{R^*} e^{-(r^*-g)x} l(x) \, dx + be^{bR^*} \int_{R^*}^{T} e^{-r^*x} l(x) \, dx}{\int_{0}^{T} e^{-px} l(x) \, dx} \right] \left( \frac{\int_{0}^{T} e^{-(g+n-r^*+\rho)x} l(x) \, dx}{\int_{0}^{R^*} e^{-nx} l(x) \, dx} \right) - 1 \]

(d) Summary: Under the restriction

\[1 - \tau - b > 0,\]

the new 3 conditions are: (37), (33) and (32).